MATH.MA.210 Discrete mathematics Exam, 08.05.2025

You are to answer **five** out of the following six questions. Do not submit more answers, I will only grade the first 5, in the order in which they appear on the paper.

(Those that have agreed on a special arrangement for the exam should answer **four**, to give you more time per question)

Questions

1. The universal set in this question is the *positive integers*, i.e., integers greater than or equal to 1. You will be asked to define sets or write logical statements or predicates. Please carefully think which is which. You are allowed to use single digit numbers 1–9 as constants in your formulas.

   (a) Define the set of even numbers

   (b) Write the predicate $p(x)$ which states that x is a prime number

   (c) Write the *Goldbach conjecture*, which says "Every even number can be expressed as a sum of two primes".

2. Prove the following is true for $n \in \mathbb{Z}_+$ by induction:

$$5^n - 1 \text{ is divisible by } 4$$

   NOTE: Use induction!

3. The natural sciences and engineering programme has 100 students. 60 study mathematics. 70 study physics. 50 study Chemistry. 30 students study both Math and Physics, while 25 study both Math and Chemistry. 35 study both physics and chemistry. 10 study all three subjects.

   (a) How many students study physics and chemistry but not mathematics?

   (b) How many students study *exactly one of the three topics*?

   (c) How many students study at least two of the three topics?

4. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ be a set and define the relation R in such a way that $(a, b) \in R$ (or, if we prefer the infix notation, aRb) if and only if a divides b

   (a) Is R an equivalence relation? Prove or explain why not.

   (b) Is R a partial order relation? Prove or explain why not

   (c) An element x is said to be *maximal* element of a relation R if and only if there does not exist an element $y \neq x$ such that $(x, y) \in R$, and similarly, *minimal* if and only there does not exist an element $y \neq x$ such that $(y, x) \in R$. Find the maximal and minimal elements of R in this case

5. A six-sided die is put on the table in such a way that the number 1 is on top. One of the numbers is facing you. (Remember, a 6-sided die has numbers 1–6 and the sum of opposite sides is always 7) You are allowed to do one of the following operations on the dice:

- Rotate the die without lifting it off the table, i.e., the 1 remains on top all the time. There are four ways of doing this, but one of them rotates the die 360 degress, so it doesn't actually do anything.
- Flip the die so that the number on top becomes the number on the bottom and vice versa. (I.e., if the top number is 1 then it becomes 6, and if it is 6 it becomes 1) while the number facing you remains the same.

(a) How many different configurations are possible for the die?

(b) Show that these rotations and flips acting on the die form a group.

(c) Is the group commutative? Either prove that it is or show a situation where it is not.

6. (a) What is the remainder when 3^{2023} is divided by 14? HINT: Euler is your friend, but if you like hard work, you can still do without him.

(b) Which of the following equations have unique solutions, and why/why not? List all solutions (i.e., all equivalence classes, obviously, as there are infinite many solutions when a solution exists)

i. $4x \equiv 7 \bmod 14$

ii. $4x \equiv 8 \bmod 14$

APPENDIX: Some helpful formulas and definitions

Definition 1. If a relation is reflexive, symmetric, and transitive, we call it an *equivalence relation* or simply an *equivalence.*

Definition 2. If a relation is reflexive, antisymmetric, and transitive, we call it an *partial order relation.*

Definition 3. A *group* is a pair $(G, \bullet)$ where G is a set $\bullet$ is an operation on G with the following properties

1. Associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for a, b, and $c \in G$.
2. Identity element: There exists an element $e \in G$ such that $e \bullet a = a \bullet e = a$ for every $a \in G$.
3. Inverse: for every $a \in G$ there is an element $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

Definition 4. Let $n \in \mathbb{Z}_+$. If for two numbers $a, b \in \mathbb{Z}$ we have $n \mid (a - b)$, we say that a is *congruent* with b *modulo* n and we denote $a \equiv b \pmod{n}$ or $a \equiv_n b$.