MATH.MA.210 Discrete mathematics Exam, 23.09.2024

Answer **five** questions out of the following six.

(Those with special arrangement answer **four** questions)

All questions have the same value.

Questions

1. We define the following propositions:
   p means "It is sunny", q means "It is a windy", and r means "It is cold".
   Also we define the following predicates:
   $A(x)$ means person x goes sailing. $B(x)$ means the person x goes swimming. $C(x)$ means x gets a tan.
   The universal set is some set of people, that includes at least the person "Jack".
   (a) Write the following sentence as a formula: "If it is sunny, everyone who goes swimming will get a tan."
   (b) Write the following sentence as a formula: "If it is windy, Jack goes sailing"
   (c) Write the following sentence as a formula: "If there is someone who goes sailing and gets a tan, then it is sunny and not cold".

2. Prove the following is true for $n \in \mathbb{Z}_+$ using induction:
$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

3. Among 50 patients admitted to a hospital, 25 are diagnosed with pneumonia, 30 with bronchitis, and 10 with both pneumonia and bronchitis. Determine:
   (a) The number of patients diagnosed with pneumonia or bronchitis (or both).
   (b) The number of patients not diagnosed with pneumonia or bronchitis.

4. Let A and B be sets such that $|A| = m$ and $|B| = n$, and assume that $m \leq n$.
   (a) How many relations exist between A and B?
   (b) How many of these relations are functions?
   (c) Assuming $A = B$, how many functions are bijections?

5. Let the operation $\star$ be defined for *odd integers* $\mathbb{Z}^{odd}$ such that
$$x \star y = x + y - 1$$
   Prove that $\mathbb{Z}^{odd}$ with the operation $\star$ forms a group, and explain what is the identity element of the group, and how to calculate the inverse for each element.

6. (a) What is the remainder when 4^{119} is divided by 5?
   (b) Which of the following equations have unique solutions, and why/why not? List all solutions (i.e., all equivalence classes, obviously, as there are infinite many solutions when a solution exists)
      i. $4x \equiv 7 \bmod 12$
      ii. $4x \equiv 7 \bmod 11$
      iii. $2x \equiv 6 \bmod 12$

APPENDIX: Some helpful formulas and definitions

Definition 1. If a relation is reflexive, symmetric, and transitive, we call it an *equivalence relation* or simply an *equivalence.*

Definition 2. A *group* is a pair $(G, \bullet)$ where G is a set $\bullet$ is an operation on G with the following properties

1. Associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for a, b, and $c \in G$.
2. Identity element: There exists an element $e \in G$ such that $e \bullet a = a \bullet e = a$ for every $a \in G$.
3. Inverse: for every $a \in G$ there is an element $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

Definition 3. Let $n \in \mathbb{Z}_+$. If for two numbers $a, b \in \mathbb{Z}$ we have $n \mid (a - b)$, we say that a is *congruent* with b *modulo* n and we denote $a \equiv b \pmod{n}$ or $a \equiv_n b$.