# TTA-45046 Financial Engineering

**Exam**
May 7, 2019

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This is a closed-book exam, a calculator allowed. Please answer in English. Good luck!

**Question 1.**

a) Explain Swaptions (2 p)

b) Explain Shout Option (2 p)

c) Let $\eta$ be annual return that satisfies $S_T = S_0 e^{\eta T}$, $T > 0$. Express the mean and standard deviation of $\eta$. What happens to the standard deviation of $\eta$ if the length of the investment period $T$ increases? What is your financial interpretation on that? (2 p)

**Question 2.**

a) Assume that the stock prices follow

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Apply Ito's lemma to derive the process of $f_t = 1/S_t$. [Remember that $D(1/g(x)) = -g'(x)/(g(x)^2)$]. (3 p)

b) Stochastic volatility models can be estimated (calibrated) using data on the cross-sections of options on a give day(s) or using time-series data on an underying stock. Compare these data sources and estimation approaches in terms of their advantages and disadvantages. (3 p)

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**Question 3.** Consider a *Up-and-Out Binary (Cash-or-Nothing) Put* European Option that ceases to exist if the asset price, $\{S_t; 0 \leq t \leq T\}$ reaches a barrier $H > S_0$. Otherwise it pays \$1 if $S_T \leq K$ and zero if $S_T > K$ at the maturity $T$. Here $S_0$ is the current stock price, $S_T$ the terminal stock price, and $K$ the strike price. Suppose that the risk-free interest rate, $r$, and volatility, $\sigma$, are constants, and that the stock price follows geometric Brownian motion.

Give a pseudo code that prices the above contract using Monte Carlo methods with *antithetic* variates. (6 p)

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## Appendix

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- **Standard normal distribution**: A standard normal distribution is a normal distribution with zero mean and unit variance, given by the probability density function (here $n(x) \equiv N'(x)$, where $N(x)$ is the cumulative standard normal distribution function)

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

- **Zero-coupon bond (discount bond)**: The value of a T-maturity zero-coupon bond at time $t$ is

$$D(t,T) = \exp\left(-\int_t^T r_s ds\right),$$

where $r$ is a non-stochastic instantaneous interest rate (short rate). If $r$ is stochastic, then

$$D(t,T) = \mathbb{E}_t\left\{\exp\left(-\int_t^T r_s ds\right)\right\}.$$

- **The binomial model**: Suppose that the stock is worth $S_0$ today and either $US_0$ or $DS_0$ after time $\Delta t$, where $U > 1 > D > 0$ denote up- and down-movements. The risk-neutral probability of an increase in stock price is expressed as

$$p = \frac{e^{r\Delta t} - D}{U - D}.$$

- **The dynamics of the risk-free asset**: Suppose that $r > 0$ is the instantaneous risk-free interest rate. Then the dynamics of an asset that earns rate $r$ can be expressed as

$$dB_t = rB_t dt, \quad B_0 > 0$$

where $B_0 > 0$.

- **Wiener process**: In continuous time, we write

$$dW_t = \epsilon_t \sqrt{dt},$$

and in discrete time

$$\Delta W_t = \epsilon_t \sqrt{\Delta t}.$$

- **The dynamics of a risky asset under geometric Brownian motion**: Suppose thet $\mu \in \mathbb{R}$ is the expected price appreciation and $\sigma > 0$ the instantaneous volatility, $\epsilon$ i.i.d. standard normal random variable, and $S_0 > 0$ the initial stock price. If the stock price is assumed to evolve according to geometric Brownian motion, we can write

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s, s > 0. \tag{1}$$

- **Itô's lemma**: Suppose that stock price follows Eq. (1). If $F$ is a function of stock price $S$ and time $t$, then $f_t = F(t, S(t))$ satisfies the Itô equation

$$\begin{aligned} df_t = & \left\{ \frac{\partial F}{\partial t}(t, S_t) + \mu S_t \frac{\partial F}{\partial s}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t) \right\} dt \\ & + \sigma S_t \frac{\partial F}{\partial s}(t, S_t) dW_t. \end{aligned}$$

- **Analytical solution for the future stock price under geometric Brownian motion**: Suppose that $S_0 > 0$. Then

$$S_T = S_0 \exp\left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) T + \sigma W(T) \right\}.$$

- **Black-Scholes equation**: Suppose that $C(t, S(t))$ is the price of a European-type derivative asset written on stock $S(t)$. Then, with the assumptions of the Black-Scholes model, the price of the derivative asset satisfies the following differential equation whenever $C$ is twice differentiable with respect to $S$ and once with respect to $t$:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial s^2} = rC.$$

- **Black-Scholes formula**: With the assumptions of the Black-Scholes model, the solution for a call option is

$$C(t, s) = se^{-q(T-t)} N(d_1(t, s)) - Ke^{-r(T-t)} N(d_2(t, s)),$$

with $N(x)$ denoting cumulative standard normal distribution and

$$\begin{aligned} d_1(t, s) &= \frac{\ln s - \ln K + (r - q + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \\ d_2(t, s) &= \frac{\ln s - \ln K + (r - q - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = d_1(t, s) - \sigma\sqrt{T - t}. \end{aligned}$$

- **Greeks:**

$$\Delta = \frac{\partial C}{\partial s} = e^{-q(T-t)}N(d_1),$$
$$\Gamma = \frac{\partial^2 C}{\partial s^2} = \frac{N'(d_1)e^{-q(T-t)}}{s\sigma\sqrt{T-t}}.$$
$$\Theta = \frac{\partial C}{\partial t} = -\frac{sN'(d_1)\sigma e^{-q(T-t)}}{2\sqrt{T-t}} + qsN(d_1)e^{-q(T-t)} - rKe^{-r(T-t)}N(d_2).$$
$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = s\sqrt{T-t}N'(d_1)e^{-q(T-t)}.$$
$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)$$

- **Pricing under martingale measure**: Let $X$ be as the numeraire (it can be a stock, the risk-free asset or any tradable asset). Moreover, $\Pi_t$ is the price of a derivative security at time $t$ that is priced according to the formula
$$\Pi_t = X_t \mathbb{E}_t^Q \left[\frac{\Pi_T}{X_T}\right],$$
where $T > t$ and $\mathbb{Q}$ is the martingale measure with $X$ as the numeraire.

- **Girsanov's theorem**: Let $W_t$ be a Brownian motion under probability measure $\mathbb{P}$. If for any (stochastic process) $\upsilon_t$
$$\int_0^t \upsilon_s^2 ds < \infty$$
with probability one. Moreover, by determining $\mathbb{Q}$ by its Radon-Nikodym derivative with respect to the original measure $\mathbb{P}$, $\mathbb{E}_t(\xi(t)X) = \mathbb{E}_t^Q(X)$, where $\xi(t) = d\mathbb{Q}/d\mathbb{P}$,
$$\xi(t) = \exp\left\{\int_0^t \upsilon_s dW_s - \frac{1}{2}\int_0^t \upsilon_s^2 ds\right\}.$$
Then under the equivalent probability measure $\mathbb{Q}$,
$$W_t^Q = W_t - \int_0^t \upsilon_s ds$$
is also a Brownian motion. The last equation can be rewritten as
$$dW_t = dW_t^Q + \upsilon_t dt.$$

- **Currency derivatives**: Suppose that the spot exchange rate at time $t$ is denoted by $X_t$ and that it follows geometric Browniain motion:
$$dX_t = X_t\alpha dt + X_t\sigma dW_t, X_0 = x, x > 0,$$
$$dB_{b,t} = r_b B_{b,t} dt$$
$$dB_{q,t} = r_q B_{q,t} dt,$$

where the holder of the base currency earns the rate of $r_b$ and the holder of the quote currency earns $r_q$ and consider a call option which gives the owner of the option to buy one unit of the base currency (say euros) at the price $K$ in the quote currency (say dollars). Then, with the Black-Scholes assumptions, the price of a call is given by

$$\begin{aligned} H(t,x) &= xe^{-r_b(T-t)}N(d_1(t,x)) - Ke^{-r_q(T-t)}N(d_2(t,x)), \\ d_1(t,x) &= \frac{\ln(x/K) + \left(r_q - r_b + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2(t,x) &= \frac{\ln(x/K) + \left(r_q - r_b - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

where $\sigma$ is the instantaneous volatility of $x$.

- **Cash-or-nothing option**: The Black-Scholes price of a cash-or-nothing option is given by

$$\begin{aligned} C_{cn}(t,s) &= e^{-r(T-t)}N(d_2(t,s)), \\ d_2 &= \frac{\ln s - \ln K + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

- **Geometric average Asian option**: Suppose that Black-Sholes assumptions hold. Then the price of a geometric average price call is given by

$$\begin{aligned} C_{avp}^{geom}(t,s) &= e^{-rT}\left(se^{\frac{1}{2}(r-\sigma^2/6)T}N(d_1(t,s)) - KN(d_2(t,s))\right), \\ d_1(t,s) &= \frac{\ln(s/K) + \left(\frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right) + \frac{1}{6}\sigma^2\right)T}{\frac{\sigma}{\sqrt{3}}\sqrt{T}}, \\ d_2(t,s) &= \frac{\ln(s/K) + \left(\frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right) - \frac{1}{6}\sigma^2\right)T}{\frac{\sigma}{\sqrt{3}}\sqrt{T}}. \end{aligned}$$

- **Two correlated random variables**: Suppose that the correlation between random variables $\varepsilon_1$ and $\varepsilon_2$ is $\rho_{12}$. Then we can write

$$\varepsilon_2 = \rho_{12}\varepsilon_1 + \sqrt{1-\rho_{12}^2}e_{12},$$

where $e_{12}$ is $N(0,1)$ and independent of $\varepsilon_1$.

- **Forward rates**: Let $F(t;T_i,T_j)$ be the forward rate from time $T_i$ to time $T_j$ at time $t$, $T_j > T_i > t > 0$, and $D(t,T_j)$ the price of a $T_j$ maturity zero-coupon bond (discount bond) at time $t$. Moreover, let $\tau_{ij}$ be a chosen time measure between $T_i$ and $T_j$. Then the following relation holds:

$$F(t;T_i,T_j) = \frac{1}{\tau_{ij}}\left(\frac{D(t,T_i)}{D(t,T_j)} - 1\right).$$

- **The legs of interest rate swaps**: For simplicity, we denote $F_{i,i+1}(t)$ as $F_i(t)$, and $D(t, T_i)$ as $D_i(t)$. The present value of a cash flow on the fixed leg is

$$V_j^{\text{fix}} = K\tau_j D_{j+1}(t),$$

where $K$ is the pre-agreed swap rate. The present value of a cashflow on the floating leg is

$$\begin{aligned} V_j^{\text{flo}} =& \tau_j L_j D_{j+1}(t) \\ &= \tau_j F_j(t) D_{j+1}(t) \\ &= D_j(t) - D_{j+1}(t). \end{aligned}$$

- **The market swap rate**: The market swap rate is a rate that makes the interest rate swap a fair contract at the present time, and it can be expressed as

$$X_{nN}(t) = \sum_{j=n}^{N-1} w_j(t) F_j(t),$$

where

$$w_j(t) = \frac{\tau_j D_{j+1}(t)}{\sum_{i=0}^{N-1} \tau_i D_{i+1}(t)}$$

or alternativelty as

$$X_{nN}(t) = \frac{D_n(t) - D_N(t)}{\sum_{j=n}^{N-1} \tau_j D_{j+1}(t)}.$$

- **The Black formula**: The Black-Scholes value of the caplet is

$$\begin{aligned} \mathbf{CAP}_i(F_i(0)) &= \tau_i D_{i+1}(0) \mathbb{E}_0^{i+1}\left[(F_i(T_i) - K)^+\right] \\ &= \tau_i D_{i+1}(0)\left[F_i(0)N(d_1) - KN(d_2)\right], \\ d_1(F_i(0)) &= \frac{\ln(F_i(0)/K) + \frac{1}{2}\sigma_i^2 T_i}{\sigma_i\sqrt{T_i}}, \\ d_2(F_i(0)) &= \frac{\ln(F_i(0)/K) - \frac{1}{2}\sigma_i^2 T_i}{\sigma_i\sqrt{T_i}}. \end{aligned}$$

- **Swaptions**: The Black-Sholes price of a swaption is given by

$$\begin{aligned} \mathbf{PS}_{nN}(X_{nN}(0)) &= B_{nN}(0)\mathbb{E}_0^{nN}\left[X_{nN}(T_n) - K\right] \\ &= B_{nN}(0)\left[X_{nN}(0)N(d_1) - KN(d_2)\right], \\ d_1(X_{nN}(0)) &= \frac{\ln(X_{nN}(0)/K) + \frac{1}{2}\sigma_{nN}^2 T_n}{\sigma_{nN}\sqrt{T_n}}, \\ d_2(X_{nN}(0)) &= \frac{\ln(X_{nN}(0)/K) - \frac{1}{2}\sigma_{nN}^2 T_n}{\sigma_{nN}\sqrt{T_n}}. \end{aligned}$$