## TTA-45036 Introduction to Financial Engineering and Derivatives Markets Appendix

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• Standard normal distribution: A standard normal distribution is a normal distribution with zero mean and unit variance, given by the probability density function (here  $n(x) \equiv N'(x)$ , where N(x) is the cumulative standard normal distribution function)

$$n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

• **Zero-coupon bond (discount bond)**: The value of a T-maturity zero-coupon bond at time t is

$$D(t,T) = \exp\left(-\int_{t}^{T} r_{s} ds\right),\,$$

where r is a non-stochastic instantaneous interest rate (short rate). If r is stochastic, then

$$D(t,T) = \mathbb{E}_t \left\{ \exp\left(-\int_t^T r_s ds\right) \right\}.$$

• The binomial model: Suppose that the stock is worth  $S_0$  today and either  $US_0$  or  $DS_0$  after time  $\Delta t$ , where U>1>D>0 denote upand down-movements. The risk-neutral probability of an increase in stock price is expressed as

$$p = \frac{e^{r\Delta t} - D}{U - D}.$$

• The dynamics of the risk-free asset: Suppose that r > 0 is the instantaneous risk-free interest rate. Then the dynamics of an asset that earns rate r can be expressed as

$$dB_t = rB_t dt$$
,  $B_0 > 0$ 

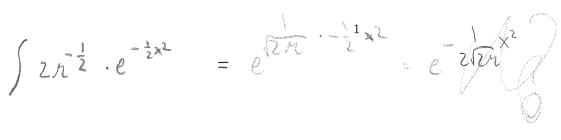
where  $B_0 > 0$ .

• Wiener process: In continuous time, we write

$$dW_t = \epsilon_t \sqrt{dt},$$

and in discrete time

$$\Delta W_t = \epsilon_t \sqrt{\Delta t}.$$



• The dynamics of a risky asset under geometric Brownian motion: Suppose that  $\mu \in \mathbb{R}$  is the expected price appreciation and  $\sigma > 0$  the instantaneous volatility,  $\epsilon$  i.i.d. standard normal random variable, and  $S_0 > 0$  the initial stock price. If the stock price is assumed to evolve according to geometric Brownian motion, we can write

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s, s > 0 \tag{1}$$

where

$$dW_t = \epsilon_t \sqrt{dt}.$$

• Analytical solution for the future stock price under geometric Brownian motion: Suppose that  $S_0 > 0$ . Then

$$S_T = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\epsilon\right\}.$$

• Black-Scholes equation: Suppose that C(t, S(t)) is the price of a European-type derivative asset written on stock S(t). Then, with the assumptions of the Black-Scholes model, the price of the derivative asset satisfies the following differential equation whenever C is twice differentiable with respect to S and once with respect to t:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial s^2} = rC.$$

• Black-Scholes formula: With the assumptions of the Black-Scholes model, the solution for a European call option is

$$C(t,s) = se^{-q(T-t)}N(d_1(t,s)) - Ke^{-r(T-t)}N(d_2(t,s)),$$

with N(x) denoting cumulative standard normal distribution and

$$d_1(t,s) = \frac{\ln s - \ln K + (r - q + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_2(t,s) = \frac{\ln s - \ln K + (r - q - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = d_1(t,s) - \sigma\sqrt{T - t}.$$

• Greeks:

$$\Delta = \frac{\partial C}{\partial s} = e^{-q(T-t)}N(d_1),$$

$$\Gamma = \frac{\partial^2 C}{\partial s^2} = \frac{N'(d_1)e^{-q(T-t)}}{s\sigma\sqrt{T-t}}.$$

$$\Theta = \frac{\partial C}{\partial t} = -\frac{sN'(d_1)\sigma e^{-q(T-t)}}{2\sqrt{T-t}} + qsN(d_1)e^{-q(T-t)} - rKe^{-r(T-t)}N(d_2).$$

$$V = \frac{\partial C}{\partial \sigma} = s\sqrt{T-t}N'(d_1)e^{-q(T-t)}.$$

$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)$$