TTA-45036 Introduction to Financial Engineering and Derivatives Markets
Appendix

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- **Standard normal distribution**: A standard normal distribution is a normal distribution with zero mean and unit variance, given by the probability density function (here $n(x) \equiv N'(x)$, where $N(x)$ is the cumulative standard normal distribution function)

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

- **Zero-coupon bond (discount bond)**: The value of a T-maturity zero-coupon bond at time t is

$$D(t,T) = \exp\left(-\int_t^T r_s ds\right),$$

where r is a non-stochastic instantaneous interest rate (short rate). If r is stochastic, then

$$D(t,T) = \mathbb{E}_t\left\{\exp\left(-\int_t^T r_s ds\right)\right\}.$$

- **The binomial model**: Suppose that the stock is worth S_0 today and either US_0 or DS_0 after time Δt, where $U > 1 > D > 0$ denote up- and down-movements. The risk-neutral probability of an increase in stock price is expressed as

$$p = \frac{e^{r\Delta t} - D}{U - D}.$$

- **The dynamics of the risk-free asset**: Suppose that $r > 0$ is the instantaneous risk-free interest rate. Then the dynamics of an asset that earns rate r can be expressed as

$$dB_t = rB_t dt, \quad B_0 > 0$$

where $B_0 > 0$.

- **Wiener process**: In continuous time, we write

$$dW_t = \epsilon_t\sqrt{dt},$$

and in discrete time

$$\Delta W_t = \epsilon_t\sqrt{\Delta t}.$$

$$\int 2\pi^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}x^2} = e^{\frac{1}{\sqrt{2\pi}} \cdot -\frac{1}{2}x^2} = e^{-\frac{1}{2\sqrt{2\pi}}x^2}$$

$d(e^x) \cdot$

- **The dynamics of a risky asset under geometric Brownian motion**: Suppose thet $\mu \in \mathbb{R}$ is the expected price appreciation and $\sigma > 0$ the instantaneous volatility, ϵ i.i.d. standard normal random variable, and $S_0 > 0$ the initial stock price. If the stock price is assumed to evolve according to geometric Brownian motion, we can write

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s, s > 0 \tag{1}$$

where

$$dW_t = \epsilon_t \sqrt{dt}.$$

- **Analytical solution for the future stock price under geometric Brownian motion**: Suppose that $S_0 > 0$. Then

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\epsilon \right\}.$$

- **Black-Scholes equation**: Suppose that $C(t, S(t))$ is the price of a European-type derivative asset written on stock $S(t)$. Then, with the assumptions of the Black-Scholes model, the price of the derivative asset satisfies the following differential equation whenever C is twice differentiable with respect to S and once with respect to t:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial s^2} = rC.$$

- **Black-Scholes formula**: With the assumptions of the Black-Scholes model, the solution for a European call option is

$$C(t, s) = se^{-q(T-t)}N(d_1(t, s)) - Ke^{-r(T-t)}N(d_2(t, s)),$$

with $N(x)$ denoting cumulative standard normal distribution and

$$\begin{aligned} d_1(t, s) &= \frac{\ln s - \ln K + (r - q + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \\ d_2(t, s) &= \frac{\ln s - \ln K + (r - q - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = d_1(t, s) - \sigma\sqrt{T - t}. \end{aligned}$$

- **Greeks:**

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial s} = e^{-q(T-t)}N(d_1), \\ \Gamma &= \frac{\partial^2 C}{\partial s^2} = \frac{N'(d_1)e^{-q(T-t)}}{s\sigma\sqrt{T - t}}. \\ \Theta &= \frac{\partial C}{\partial t} = -\frac{sN'(d_1)\sigma e^{-q(T-t)}}{2\sqrt{T - t}} + qsN(d_1)e^{-q(T-t)} - rKe^{-r(T-t)}N(d_2). \\ \mathcal{V} &= \frac{\partial C}{\partial \sigma} = s\sqrt{T - t}N'(d_1)e^{-q(T-t)}. \\ \rho &= \frac{\partial C}{\partial r} = K(T - t)e^{-r(T-t)}N(d_2) \end{aligned}$$