SGN-11007 Introduction to Signal Processing,
Final Exam, 17.10.2019,
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- Own calculators can be used in the exam.
- You may take the examination paper with you.

1. (a) Analog signal consists of a single sine wave with the frequency 500 Hz. A system samples the signal at intervals of $T = \frac{1}{500}$ s.
   i. What is the Nyquist frequency? (1p)
   ii. Does aliasing happen? (1p)
   iii. What is the observed frequency of the sine wave in the digital signal after the sampling? (1p)

   (b) Calculate the DFT of the sequence $x(n) = (-1, 0, 2, 3, 0, 0, -3, 3)$ using the FFT algorithm. You can skip part of the calculations by utilizing this information: the DFT of the sequence $(-1, 2, 0, -3)$ is $(-2, -1 - 5i, 0, -1 + 5i)$ and the DFT of $(0, 3, 0, 3)$ is $(6, 0, -6, 0)$. (3p)

2. An infinitely long impulse response of a filter is shown in Figure 1.

   (a) Is the filter an FIR or IIR filter? Justify. (2p)
   (b) Give the transfer function of the filter. (2p)
   (c) Is the filter stable? Why / why not? (2p)

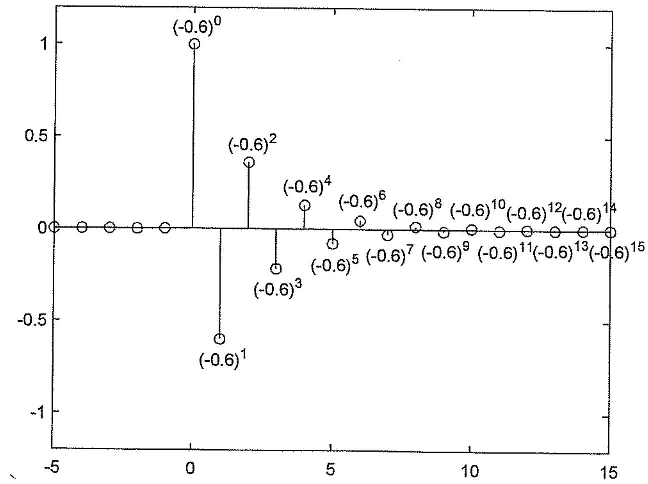

Figure 1: Impulse response of Task 2.

3. The signal $x(n)$ with the sampling rate 25 kHz should be converted to a signal with the sampling rate 15 kHz. Determine the steps of the conversion as a block diagram using resampling ($\boxed{\uparrow L}$ and $\boxed{\downarrow M}$) and low-pass filtering ($\boxed{H(z)}$). Specify the passband and stopband intervals of the required low-pass filters in normalized frequencies, when the frequencies on the interval $0 - 3.75$ kHz are to be preserved. (6p)

4. Design using the window design method a filter (i.e. find out its impulse response) satisfying the following requirements:

Stopband	[12 kHz, 16 kHz]
Passband	[0 kHz, 10 kHz]
Passband ripple	0.5 dB
Minimum stopband attenuation	20 dB
Sampling frequency	30 kHz

Use the tables below. (6p)

5. (a) When designing an LDA classifier for the data in Figure 2, the covariance matrices of the classes were found to be:

$$\mathbf{C}_1 = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \mathbf{C}_2 = \frac{1}{3}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}.$$

What is the vector $\mathbf{w}$ for this LDA classifier? (4p)

(b) What is the class of the sample $(-3, -4)$ when you classify it with the LDA classifier. Justify the selection of the class. (2p)

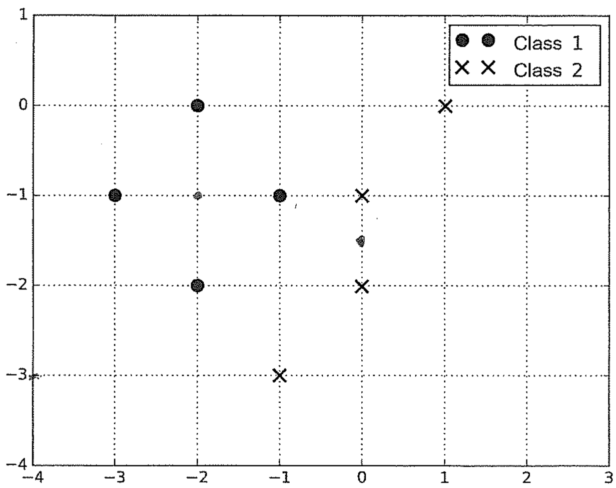

Figure 2: Material of Task 5.

Tables

Ideal filter type	Impulse response when $n \neq 0$	$n = 0$
Low-pass	$2f_c \mathrm{sinc}(n \cdot 2\pi f_c)$	$2f_c$
High-pass	$-2f_c \mathrm{sinc}(n \cdot 2\pi f_c)$	$1 - 2f_c$
Band-pass	$2f_2 \mathrm{sinc}(n \cdot 2\pi f_2) - 2f_1 \mathrm{sinc}(n \cdot 2\pi f_1)$	$2(f_2 - f_1)$
Band-stop	$2f_1 \mathrm{sinc}(n \cdot 2\pi f_1) - 2f_2 \mathrm{sinc}(n \cdot 2\pi f_2)$	$1 - 2(f_2 - f_1)$

Name of the window function	Transition bandwidth (normalized)	Passband ripple (dB)	Minimum stopband attenuation (dB)	Window expression $w(n)$, when $\lvert n\rvert \leq (N-1)/2$
Rectangular	0.9/N	0.7416	21	1
Bartlett	3.05/N	0.4752	25	$1-\frac{2\lvert n\rvert}{N-1}$
Hanning	3.1/N	0.0546	44	$0.5+0.5\cos\left(\frac{2\pi n}{N}\right)$
Hamming	3.3/N	0.0194	53	$0.54+0.46\cos\left(\frac{2\pi n}{N}\right)$
Blackman	5.5/N	0.0017	74	$0.42+0.5\cos\left(\frac{2\pi n}{N}\right)+0.08\cos\left(\frac{4\pi n}{N}\right)$

Equations

$$ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\begin{cases} X(n)=X_0(n)+w_N^{-n}X_1(n), & \text{when } n=0,1,2,\ldots,N/2-1 \\ X(n)=X_0(n-N/2)+w_N^{-n}X_1(n-N/2), & \text{when } n=N/2,N/2+1,\ldots,N-1 \end{cases}$$

Some Wikipedia pages that might be useful

Suppose two classes of observations have means $\vec{\mu}_0, \vec{\mu}_1$ and covariances Σ_0, Σ_1. Then the linear combination of features $\vec{w}\cdot\vec{x}$ will have means $\vec{w}\cdot\vec{\mu}_i$ and variances $\vec{w}^T\Sigma_i\vec{w}$ for $i=0,1$. Fisher defined the separation between these two distributions to be the ratio of the variance between the classes to the variance within the classes:

$$S=\frac{\sigma^2_{\text{between}}}{\sigma^2_{\text{within}}}=\frac{(\vec{w}\cdot\vec{\mu}_1-\vec{w}\cdot\vec{\mu}_0)^2}{\vec{w}^T\Sigma_1\vec{w}+\vec{w}^T\Sigma_0\vec{w}}=\frac{(\vec{w}\cdot(\vec{\mu}_1-\vec{\mu}_0))^2}{\vec{w}^T(\Sigma_0+\Sigma_1)\vec{w}}$$

This measure is, in some sense, a measure of the signal-to-noise ratio for the class labelling. It can be shown that the maximum separation occurs when

$$\vec{w}\propto(\Sigma_0+\Sigma_1)^{-1}(\vec{\mu}_1-\vec{\mu}_0)$$

When the assumptions of LDA are satisfied, the above equation is equivalent to LDA.

Be sure to note that the vector $\vec{w}$ is the normal to the discriminant hyperplane. As an example, in a two dimensional problem, the line that best divides the two groups is perpendicular to $\vec{w}$.

Generally, the data points to be discriminated are projected onto $\vec{w}$; then the threshold that best separates the data is chosen from analysis of the one-dimensional distribution. There is no general rule for the threshold. However, if projections of points from both classes exhibit approximately the same distributions, a good choice would be the hyperplane between projections of the two means, $\vec{w}\cdot\vec{\mu}_0$ and $\vec{w}\cdot\vec{\mu}_1$. In this case the parameter c in threshold condition $\vec{w}\cdot\vec{x}>c$ can be found explicitly:

$$c=\vec{w}\cdot\frac{1}{2}(\vec{\mu}_0+\vec{\mu}_1)=\frac{1}{2}\vec{\mu}_1^T\Sigma_1^{-1}\vec{\mu}_1-\frac{1}{2}\vec{\mu}_0^T\Sigma_0^{-1}\vec{\mu}_0.$$

A more condensed form of the difference equation is:

$$y[n] = \frac{1}{a_0}\left(\sum_{i=0}^{P} b_i x[n-i] - \sum_{j=1}^{Q} a_j y[n-j]\right)$$

which, when rearranged, becomes:

$$\sum_{j=0}^{Q} a_j y[n-j] = \sum_{i=0}^{P} b_i x[n-i]$$

To find the transfer function of the filter, we first take the Z-transform of each side of the above equation, where we use the time-shift property to obtain:

$$\sum_{j=0}^{Q} a_j z^{-j} Y(z) = \sum_{i=0}^{P} b_i z^{-i} X(z)$$

We define the transfer function to be:

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{i=0}^{P} b_i z^{-i}}{\sum_{j=0}^{Q} a_j z^{-j}} \end{aligned}$$

Considering that in most IIR filter designs coefficient a_0 is 1, the IIR filter transfer function takes the more traditional form:

$$H(z) = \frac{\sum_{i=0}^{P} b_i z^{-i}}{1 + \sum_{j=1}^{Q} a_j z^{-j}}$$

Inversion of 2 × 2 matrices [edit]

The *cofactor equation* listed above yields the following result for 2 × 2 matrices. Inversion of these matrices can be done as follows:[6]

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Techniques [edit]

Conceptual approaches to sample-rate conversion include: converting to an analog continuous signal, then re-sampling at the new rate, or calculating the values of the new samples directly from the old samples. The latter approach is more satisfactory, since it introduces less noise and distortion.[3] Two possible implementation methods are as follows:

1. If the ratio of the two sample rates is (or can be approximated by)[nb 1][4] a fixed rational number *L/M*: generate an intermediate signal by inserting *L* − 1 0s between each of the original samples. Low-pass filter this signal at half of the lower of the two rates. Select every *M*-th sample from the filtered output, to obtain the result.[5]
2. Treat the samples as geometric points and create any needed new points by interpolation. Choosing an interpolation method is a trade-off between implementation complexity and conversion quality (according to application requirements). Commonly used are: ZOH (for film/video frames), cubic (for image processing) and windowed sinc function (for audio).